Output Synchronization for Heterogeneous Networks of Non-Introspective Agents ⋆

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Abstract

In this paper we consider the output synchronization problem for heterogeneous networks of linear agents. The network's communication infrastructure provides each agent with a linear combination of its own output relative to that of neighboring agents, and it allows the agents to exchange information about their own internal observer estimates. We design decentralized controllers based on setting the control input of a single root agent to zero and letting the remaining agents synchronize to the root agent. A distinguishing feature of our work is that the agents are assumed to be *non-introspective*, meaning that they possess no knowledge about their own state or output separate from what is received via the network. We also consider the problem of regulating the agreement trajectory according to an *a priori* specified reference model. In this case we assume that some of the agents have access to their own output relative to the reference trajectory.

Key words: Synchronization; Decentralization; Multi-agent systems

1 Introduction

The problem of achieving *synchronization* among agents in a network—that is, asymptotic agreement on the agents' state or output trajectories—has received substantial attention in recent years. The essential difficulty of the synchronization problem is the lack of a central authority with the ability to control the network as a whole. Instead, each agent must implement a controller based on limited information about itself and its surroundings—typically in the form of measurements of its own state or output relative to that of neighboring agents in the network.

Much of the attention has been directed toward *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), with each agent receiving information about its own state relative to that of neighboring agents (e.g., Olfati-Saber and Murray, 2003, 2004; Olfati-Saber, Fax, and Murray, 2007; Ren, Beard, and Atkins, 2007; Ren and Atkins, 2007). Roy, Saberi, and Herlugson (2007), Tuna (2008a), and Yang, Roy, Wan, and Saberi (2011a) considered this type of problem for more general observation topologies and more complex identical agent models than previously considered. Others have studied the case where the agents receive relative information about their own partial-state output (e.g., Pogromsky and Nijmeijer, 2001; Pogromsky, Santoboni, and Nijmeijer, 2002; Tuna, 2008b; Li, Duan, Chen, and Huang, 2010). A key idea in the work of Li et al. (2010), which was expanded upon by Yang, Stoorvogel, and Saberi (2011c), is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many of the results on the synchronization problem are rooted in the seminal work of Wu and Chua (1995a,b).

1.1 Heterogeneous Networks and Output Synchronization

A limited amount of research has also been conducted on *heterogeneous* networks (i.e., networks where the agent models are non-identical). Ramírez and Femat (2007) presented a robust state-synchronization design for networks of nonlinear systems with relative degree one, where each agent implements a sufficiently strong feedback based on the difference between its own state and that of a common reference model. In the work of Xiang and Chen (2007) it is assumed that a common Lyapunov function candidate is available, which is used to analyze stability with respect to a common equilibrium point. Depending on the system, some agents may also implement feedbacks to ensure stability, based on the difference between those agents' states

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and the equilibrium point. Zhao, Hill, and Liu (2011) analyzed state synchronization in a network of nonlinear agents based on the network topology and the existence of certain time-varying matrices. Controllers can be designed based on this analysis, to the extent that the available information and actuation allows for the necessary manipulation of the network topology.

The above-cited works focus on synchronizing the agents' internal states. In heterogeneous networks, however, the physical interpretation of one agent's state may be different from that of another agent. Indeed, the agents may be governed by models of different dimensions. In this case, comparing the agents' internal states is not meaningful, and it is more natural to aim for *output synchronization*—that is, agreement on some partial-state output from each agent. Chopra and Spong (2008) focused on output synchronization for weakly minimum-phase systems of relative degree one, using a pre-feedback within each agent to create a single-integrator system with decoupled zero dynamics. Pre-feedbacks were also used by Bai, Arcak, and Wen (2011) to facilitate passivity-based designs. The authors have previously considered output synchronization for right-invertible agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of asymptotically identical agents (Yang, Saberi, Stoorvogel, and Grip, 2011b).

Kim, Shim, and Seo (2011) studied output synchronization for uncertain single-input single-output, minimum-phase systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. A similar approach was taken by Wieland, Sepulchre, and Allgöwer (2011), who showed that a necessary condition for output synchronization in heterogeneous networks is the existence of a *virtual exosystem* that produces a trajectory to which all the agents asymptotically converge. If one knows the model of an observable virtual exosystem without exponentially unstable modes, which each agent is capable of tracking, then it can be implemented within each agent and synchronized via the network. The agent can then be made to track the model with the help of a local observer estimating the agent's states.

1.2 *Introspective Versus Non-Introspective Agents*

The designs mentioned above for heterogeneous networks rely—explicitly or implicitly—on some sort of self-knowledge that is separate from the information transmitted over the network. In particular, the agents may be required to know their own state, their own output, or their own state/output relative to that of a reference trajectory. In this paper we shall refer to agents that possess this type of self-knowledge as *introspective* agents, to distinguish them from *non-introspective* agents—that is, agents that have no knowledge of their own state or output separate from what is received via the network. This distinction is significant because introspective agents have much greater freedom to manipulate their internal dynamics (e.g., through the use of pre-feedbacks) and thus change the way that they present themselves to the rest of the network. The notion of a non-introspective agent is also practically relevant; for example, two vehicles in close proximity may be able to measure their relative distance without either of them having knowledge of their absolute position.

To the authors' knowledge, the only result that solves the output synchronization problem for a well-defined class of heterogeneous networks of non-introspective agents is by Zhao, Hill, and Liu (2010). In their work, the only information available to each agent is a linear combination of outputs received over the network. However, the agents are assumed to be passive—a strict requirement that, among other things, requires the agents to be weakly minimum-phase and of relative degree one.

1.3 *Contributions of This Paper*

In this paper we consider heterogeneous networks of non-introspective linear agents that receive, via the network, a linear combination of their own output relative to that of neighboring agents. In the spirit of Li et al. (2010) we also assume that the agents can exchange relative information about their internal estimates using the network's communication infrastructure. We design decentralized controllers for achieving output synchronization under a set of straightforward assumptions about the agents and the topology of the network. A version of this design has also been presented at the 2012 *American Control Conference* (Grip, Yang, Saberi, and Stoorvogel, 2012).

Based on the output-synchronization results we also consider the slightly different problem of *regulated output synchronization*. Here, the goal is not only to achieve output synchronization, but to make the synchronization trajectory follow an *a priori* given reference. When considering this problem we assume that some of the agents are introspective in the sense that they know their own output relative to that of the reference output.

1.4 *Notation*

Given a matrix A, A' denotes its transpose and A^* denotes its conjugate transpose. We denote by $A \otimes B$ the Kronecker product between matrices A and B. When clear from the context, 0 denotes a zero matrix of appropriate dimensions.

2 Problem Formulation

We consider a network of N multiple-input multiple-output agents of the form

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{1a}$$
$$y_i = C_i x_i + D_i u_i, \tag{1b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$. Our goal is to achieve output synchronization among the agents, meaning that $\lim_{t\to\infty}(y_i - y_j) = 0$ for all $i, j \in \{1,\dots,N\}$.

The agents are non-introspective; hence, agent i does not have access to its own output y_i. The only available information comes from the network, which provides each agent with a linear combination of its own output relative to that of the other agents. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$. The topology of the network can be described by a directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}.

We shall frequently make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. This matrix is known as the *Laplacian* matrix of the digraph $\mathscr{G}$ and has the property that all the row sums are zero. In terms of the coefficients of G, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j.$$

We also assume that the agents can exchange relative information about their internal estimates using the network's communication infrastructure. Specifically, agent i is presumed to have access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} g_{ij}\eta_j,$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j as part of the controller. This variable will be specified as we proceed with the control design.

2.1 *Assumptions*

We make the following assumptions about the network topology and the individual agents.

Assumption 1 *The digraph $\mathscr{G}$ has a directed spanning tree with root agent $K \in \{1,\dots,N\}$, such that for each $i \in \{1,\dots,N\} \setminus K$,*

(1) (A_i, B_i) is stabilizable
(2) (A_i, C_i) is observable
(3) (A_i, B_i, C_i, D_i) is right-invertible
(4) (A_i, B_i, C_i, D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of A_K

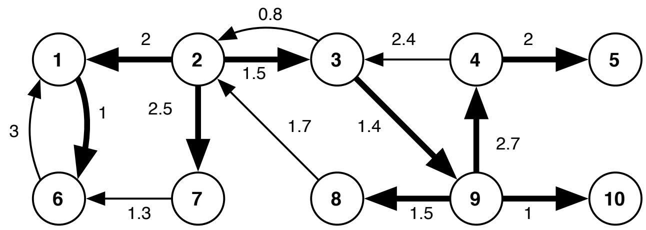

Fig. 1. The depicted digraph contains multiple directed spanning trees, rooted at nodes 2, 3, 4, 8, and 9. One of these, with root node 2, is illustrated by bold arrows.

Remark 1 A *directed tree* is a directed subgraph of $\mathscr{G}$, consisting of a subset of the nodes and edges, such that every node has exactly one parent, except a single root node with no parents. Furthermore, there must exist a directed path from the root to every other agent. A *directed spanning tree* is a directed tree that contains all the nodes of $\mathscr{G}$. A digraph may contain many directed spanning trees, and thus there may be several choices of root agent K. Fig. 1 illustrates a digraph containing multiple directed spanning trees.

Remark 2 Right-invertibility of a quadruple (A_i, B_i, C_i, D_i) means that, given a reference output $y_r(t)$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.

Let the matrix $\bar{G}_K = [g_{ij}]_{i,j\neq K}$ be defined from G by removing row and column number K, corresponding to the root of a directed spanning tree of $\mathscr{G}$. We shall need the following result, which is proven in Appendix A.

Lemma 1 *All the eigenvalues of $\bar{G}_K$ are in the open right-half complex plane.*

3 Control Design

In this section we describe the construction of decentralized controllers that achieve output synchronization. Before embarking on the actual design procedure, however, we shall describe the motivation behind the design.

The main idea is to set the control input of the root agent K to zero (i.e., $u_K = 0$) and to also set $\eta_K = 0$. We then design controllers for all the other agents such that their outputs asymptotically synchronize with the trajectory $y_K(t)$. That is, for each $i \in \{1,\dots,N\} \setminus K$ we wish to achieve $\lim_{t\to\infty}(y_i - y_K) = 0$. Equivalently, we wish to regulate the synchronization error variable

$$e_i := y_i - y_K$$

to zero, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{2a}$$

$$e_i = \begin{bmatrix} C_i & -C_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + D_i u_i. \tag{2b}$$

The system (2) is in general not stabilizable. If x_i and x_K were available to agent i as measurements, then the problem of making e_i converge to zero would nevertheless be solvable by standard output-regulation methods (see, e.g., Saberi, Stoorvogel, and Sannuti, 2000). But alas, the only information available to agent i is ζ_i and $\hat{\zeta}_i$. To achieve our objective with such limited information, we carry out our design for each agent $i \in \{1,\dots,N\} \setminus K$ in three steps.

In Step 1 we construct a new state $\bar{x}_i$, via a transformation of x_i and x_K, so that the dynamics of the synchronization error variable e_i can be described by the alternative equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{3a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i u_i := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i u_i. \tag{3b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i by removing redundant modes that have no effect on e_i. In particular, the model (2) may be unobservable, but the model (3) is always observable.

The properties of the model (3) also allow us, in Step 2 of the design, to construct a controller that regulates e_i to zero by using state feedback from $\bar{x}_i$. This controller is not directly implementable, however, because $\bar{x}_i$ is not known to agent i. This brings us to Step 3 of the design, where we construct an observer that makes an estimate of $\bar{x}_i$ available to agent i. This observer is based on the information ζ_i and $\hat{\zeta}_i$ received via the network, and it works in a distributed manner together with the observers for the other agents to achieve convergence. The observer design is based on previous results on distributed observer design for *homogeneous* networks. Since our network is heterogeneous, we first perform a second state transformation of $\bar{x}_i$ to χ_i, in order to obtain a dynamical model that is substantially the same as for the other agents. In particular, the model differences now occur only in particular locations where they can be suppressed by using high-gain observer techniques. By combining the observer estimates with the state-feedback controller designed in Step 2, we achieve output synchronization.

3.1 Design Preliminaries

Due to the design strategy of setting $u_K = 0$, the trajectory $y_K(t)$ becomes the unforced response of agent K, consisting of a linear combination of the observable modes of the pair (A_K, C_K). Asymptotically stable modes vanish as $t \to \infty$, and they therefore play no role asymptotically. For simplicity of presentation, we therefore assume that all the eigenvalues of A_K are in the closed right-half complex plane and that (A_K, C_K) is observable. We make this assumption without any loss of generality since, if A_K does contain unobservable or asymptotically stable modes, we can always create an auxiliary model excluding those modes for the purpose of control design (see Appendix C for details).

Below we describe the three steps of the design procedure that must be carried out for each agent $i \in \{1,\dots,N\} \setminus K$. In addition to agent i's system matrices (A_i, B_i, C_i, D_i), the information needed to carry out these three steps for agent i is as follows:

- the matrices A_K and C_K of the root agent
- a common integer $\bar{n}$ such that $\bar{n} \geq n_i + n_K$ for all $i \in \{1,\dots,N\} \setminus K$ [1]
- a common matrix $L \in \mathbb{R}^{p \times p\bar{n}}$, freely chosen [2]
- a common high-gain parameter $\varepsilon \in (0,1]$
- a common number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of the matrix $\bar{G}_K$ defined in Section 2.1

Based on this information, we can define the matrices $\mathscr{A} \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$, $\mathscr{C} \in \mathbb{R}^{p \times p\bar{n}}$, $\Omega_\varepsilon \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$, and $\mathscr{L}_\varepsilon \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$ as

$$\mathscr{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

$$\Omega_\varepsilon = \begin{bmatrix} I_p \varepsilon^{-1} & & \\ & \ddots & \\ & & I_p \varepsilon^{-\bar{n}} \end{bmatrix}, \quad \mathscr{L}_\varepsilon = \begin{bmatrix} 0 \\ \varepsilon^{\bar{n}+1} L \Omega_\varepsilon \end{bmatrix}.$$

The pair $(\mathscr{A} + \mathscr{L}_\varepsilon, \mathscr{C})$ is always observable; hence, we can define a matrix $\mathscr{P}_\varepsilon = \mathscr{P}_\varepsilon' > 0$ as the unique solution of the algebraic Riccati equation

$$(\mathscr{A} + \mathscr{L}_\varepsilon)\mathscr{P}_\varepsilon + \mathscr{P}_\varepsilon(\mathscr{A} + \mathscr{L}_\varepsilon)' - 2\tau \mathscr{P}_\varepsilon \mathscr{C}' \mathscr{C} \mathscr{P}_\varepsilon + I_{p\bar{n}} = 0. \tag{4}$$

The matrices $\mathscr{A}$, $\mathscr{C}$, Ω_ε, $\mathscr{L}_\varepsilon$, and $\mathscr{P}_\varepsilon$ will be used during the design procedure.

3.2 Design Procedure for Agent $i \in \{1,\dots,N\} \setminus K$

Step 1: State Transformation

Let O_i be the observability matrix corresponding to the system (2):

$$O_i = \begin{bmatrix} C_i & -C_K \\ \vdots & \vdots \\ C_i A_i^{n_i+n_K-1} & -C_K A_K^{n_i+n_K-1} \end{bmatrix}. \tag{5}$$

[1] The integer $\bar{n}$ can be defined less conservatively as a bound on $n_i + r_i$ for $i \in \{1,\dots,N\} \setminus K$, where r_i is defined during Step 1 of the design procedure for each agent.

[2] See Section 3.4 for an explanation of the purpose of L.

Let q_i denote the dimension of the null space of O_i, and define $r_i = n_K - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{n_K \times q_i}$ such that

$$O_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \operatorname{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i. \tag{6}$$

Because (A_i, C_i) and (A_K, C_K) are observable, Λ_{iu} and Φ_{iu} have full column rank (see Appendix D). Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{n_K \times n_K}$ are nonsingular. We define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} x_i - \Lambda_i M_i \Phi_i^{-1} x_K \\ -N_i \Phi_i^{-1} x_K \end{bmatrix},$$

where $M_i \in \mathbb{R}^{n_i \times n_K}$ and $N_i \in \mathbb{R}^{r_i \times n_K}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

The following lemma, which is proven in Appendix A, shows how the synchronization error e_i is given in terms of $\bar{x}_i$.

Lemma 2 *The synchronization error variable e_i is governed by dynamical equations of the form* (3)*, where $(\bar{A}_i, \bar{C}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K.*

Step 2: State-Feedback Control Design

We now design a controller as a function of $\bar{x}_i$ to regulate e_i to zero. Consider the following equations with unknowns $\Pi_i \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i \in \mathbb{R}^{m_i \times r_i}$, commonly known as the *regulator equations*:

$$\Pi_i \bar{A}_{i22} = A_i \Pi_i + \bar{A}_{i12} + B_i \Gamma_i, \tag{7a}$$
$$C_i \Pi_i - \bar{C}_{i2} + D_i \Gamma_i = 0. \tag{7b}$$

Based on Π_i and Γ_i, we define a matrix

$$\bar{F}_i = \begin{bmatrix} F_i & \Gamma_i - F_i \Pi_i \end{bmatrix}, \tag{8}$$

where F_i is chosen such that $A_i + B_i F_i$ is Hurwitz. The following lemma, which is proven in Appendix A, shows that the regulator equations (7) are always solvable and that the matrix $\bar{F}_i$ can be used to define a state-feedback controller.

Lemma 3 *The regulator equations* (7) *are solvable, and the state-feedback controller $u_i = \bar{F}_i \bar{x}_i$ ensures that* $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - y_K) = 0$.

Step 3: Observer-Based Implementation

Our last step is to design an observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. Define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is not necessarily a square matrix; however, due to observability of $(\bar{A}_i, \bar{C}_i)$, T_i is injective, which implies that $T_i' T_i$ is nonsingular. In terms of χ_i, we can write the equations governing e_i as

$$\dot{\chi}_i = (\mathscr{A} + \mathscr{L}_i)\chi_i + \mathscr{B}_i u_i, \quad \chi_i(0) = T_i \bar{x}_i(0), \tag{9a}$$
$$e_i = \mathscr{C} \chi_i + \mathscr{D}_i u_i, \tag{9b}$$

where

$$\mathscr{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathscr{B}_i = T_i \bar{B}_i, \quad \mathscr{D}_i = \bar{D}_i,$$

and where $L_i = \bar{C}_i \bar{A}_i^{\bar{n}} (T_i' T_i)^{-1} T_i'$. We construct the observer

$$\dot{\hat{\chi}}_i = (\mathscr{A} + \mathscr{L}_i)\hat{\chi}_i + \mathscr{B}_i u_i + \Omega_\varepsilon \mathscr{P}_\varepsilon \mathscr{C}'(\zeta_i - \hat{\zeta}_i), \tag{10a}$$
$$\hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i. \tag{10b}$$

Based on the observer estimate, we define the variable $\eta_i = \mathscr{C} \hat{\chi}_i + \mathscr{D}_i u_i$ to be shared with the other agents via the network's communication infrastructure as described in Section 2, and the observer-based control law

$$u_i = \bar{F}_i \hat{\bar{x}}_i. \tag{11}$$

Together, the observers for agents $i \in \{1, \ldots, N\} \setminus K$ form a distributed observer parameterized by a common high-gain parameter ε. The following lemma, which is proven in Appendix A, shows that all the observation errors vanish asymptotically if ε is chosen sufficiently small.

Lemma 4 *There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then for each $i \in \{1, \ldots, N\} \setminus K$,* $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$.

3.3 Main Result

By implementing the observer-based control law (11) for each agent $i \in \{1, \ldots, N\} \setminus K$, we obtain a decentralized controller structure that achieves output synchronization. The following theorem formalizes this result.

Theorem 1 *There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then for each $i, j \in \{1, \ldots, N\}$,* $\lim_{t\to\infty}(y_i - y_j) = 0$.

PROOF Since the systems are linear, the result follows from Lemmas 3 and 4 and the separation principle. ■

3.4 Remarks on the Design Procedure

Having presented the design procedure, some remarks are in order.

The purpose of Step 1 is to reduce the dimension of the model (2) by removing redundant modes that cannot be observed from e_i. Such modes exist if agent i and agent K share particular unforced solutions. Consider, for example, the case where agents i and K are identical. Then the states x_i and x_K cannot be individually observed from $e_i = y_i - y_K$, since there are infinitely many initial conditions that yield the unforced solution $e_i = 0$. If, on the other hand, we define the state $\bar{x}_i = x_i - x_K$, then we obtain the model $\dot{\bar{x}}_i = A_i\bar{x}_i + B_iu_i$, $e_i = C_i\bar{x}_i + D_iu_i$, which is observable. Indeed, it is easily verified that in our design procedure, identical agents yield $q_i = n_i = n_K$ and $r_i = 0$, and that $\Lambda_i = I_{n_i}$ and $\Phi_i = I_{n_K}$ are valid choices; thus, one obtains precisely $\bar{x}_i = x_i - x_K$. In the general case, Step 1 yields a model (3) that incorporates the difference between modes that are shared between agents i and K in addition to modes from both agent i and K that are not shared.

In Step 2 we must find the solutions Π_i and Γ_i of the regulator equations (7). A special situation arises when $r_i = 0$, which implies that $\bar{A}_{i22}$, $\bar{A}_{i12}$, and $\bar{C}_{i2}$ are empty matrices. In this case, Π_i and Γ_i are also empty matrices, and the need to solve the regulator equations vanishes. This situation occurs, in particular, if agent i and agent K are identical.

In Step 3, we introduce a state transformation from $\bar{x}_i$ to χ_i, where χ_i has dimension $p\bar{n}$. Since the dimension of $\bar{x}_i$ may be less than $p\bar{n}$, the transformation to χ_i may involve an over-parameterization. In this case, (9) is not the only possible dynamical model of χ_i, but it is always *one* of the possible representations. After performing the state transformation, we proceed to construct an observer that depends on a high-gain parameter ε. Following the proof of Lemma 4, it can be seen that ε must be chosen to stabilize the dynamics (A.2) by making the matrix $I_{N-1} \otimes (\mathscr{A} + \mathscr{L}_\varepsilon) - \bar{G}_K \otimes (\mathscr{P}_\varepsilon \mathscr{C}'\mathscr{C}) - \tilde{\mathscr{L}}_\varepsilon$ Hurwitz. This works because the nonzero elements of $\tilde{\mathscr{L}}_\varepsilon$ are on the form $\varepsilon^{\bar{n}+1}(L - L_i)\Omega_\varepsilon$ (meaning that $\|\tilde{\mathscr{L}}_\varepsilon\| = O(\varepsilon)$), and $\tilde{\mathscr{L}}_\varepsilon$ is therefore dominated by the Hurwitz matrix $I_{N-1} \otimes (\mathscr{A} + \mathscr{L}_\varepsilon) - \bar{G}_K \otimes (\mathscr{P}_\varepsilon \mathscr{C}'\mathscr{C})$ as $\varepsilon \to 0$. The freely chosen matrix L plays a role in determining how small ε needs to be chosen, because the difference $L - L_i$ affects the nonzero elements of $\tilde{\mathscr{L}}_\varepsilon$. If sufficient information is available about the agent models, L can be chosen to make the differences $L - L_i$ small, in order to reduce the need for high gain. If all the agents are identical, then L_i is the same for all the agents and one can choose $L = L_i$. In this case, $\tilde{\mathscr{L}}_\varepsilon$ vanishes and ε can be chosen arbitrarily. It is therefore evident that the role of ε is to suppress the differences in agent models that exist in heterogeneous networks.

3.4.1 Information Required About the Network

When designing the controller for agent i, it is necessary to know the model (A_i, B_i, C_i, D_i) of agent i, but it is not necessary to know the models of all the other agents or the exact topology of the network. Some additional information is nevertheless required, as specified in Section 3.1. To justify the required level of information, we note that most of the required information is also assumed available in the literature on homogeneous networks, albeit implicitly. In a homogeneous network, knowledge of A_i and C_i implies knowledge of A_K and C_K, since the models are identical. Moreover, $\bar{n} = 2n_i$ is a known bound on $n_i + n_K$, since the agents are of the same order. As described above, the matrices L_i are all the same in a homogeneous network; hence one can choose $L = L_i$, which means that $\varepsilon = 1$ is always a valid choice. The lower bound $\tau > 0$ on the real part of the eigenvalues of $\bar{G}_K$ can be viewed as a measure of the connectivity of the network. Similar measures of connectivity are typically assumed available in the literature on general homogeneous networks (Tuna, 2008a; Li et al., 2010; Yang et al., 2011c).

Even though exact information about the network is not required in the design process, it is nevertheless useful, as it is then possible to search for a non-conservative ε that makes $I_{N-1} \otimes (\mathscr{A} + \mathscr{L}_\varepsilon) - \bar{G}_K \otimes (\mathscr{P}_\varepsilon \mathscr{C}'\mathscr{C}) - \tilde{\mathscr{L}}_\varepsilon$ Hurwitz. One can also define τ as a tight lower bound on the real part of the eigenvalues of $\bar{G}_K$ and $\bar{n}$ as a tight bound on $n_i + r_i$ in accordance with footnote [1] on page 4.

3.5 Computational Complexity

The controllers constructed in this paper contain internal dynamics in the form of an observer for χ_i. The internal dynamics introduces additional computational complexity compared to the static control laws that have previously been used for synchronization of of single and double integrators (e.g., Olfati-Saber and Murray, 2003, 2004; Ren and Atkins, 2007) and general homogeneous networks with relative-state information (e.g., Tuna, 2008a; Yang et al., 2011a). The need for internal dynamics arises for two reasons. First, since only relative-output information is exchanged, the agents need internal observer dynamics to estimate unmeasured states. Second, since the agents are non-identical, the agreement manifold may contain modes that are not contained within all the agents, and which must therefore be replicated by internal dynamics according to the internal model principle.

The order of the internal dynamics is $\bar{n}$, which is an upper bound on $n_i + n_K$ for $i \in \{1, \ldots, N\} \setminus K$. Alternatively, as remarked in footnote [1] on page 4, $\bar{n}$ can be defined less conservatively as a bound on $n_i + r_i$. The integer r_i can be viewed as representing the order of the part of the root agent dynamics that is not contained within agent i. Hence, the computational complexity is in this case dependent on how similar the agents are to one another. Indeed, in the case of identical agents, one always has $r_i = 0$, so $\bar{n} = n_i$, meaning that each agent implements an observer of order equal to that of its own dynamics.

An interesting topic of future work is the reduction of computational complexity by finding ways to reduce the order of the internal dynamics within each agent.

4 Regulated Output Synchronization

Our focus so far has been on achieving agreement on a common output trajectory, without regard to the particular properties of that trajectory. In this section we consider the related problem of regulating the outputs toward a desired reference trajectory $y_r(t)$, which is defined as the output of an autonomous exosystem

$$\dot{\omega} = S\omega, \tag{12a}$$
$$y_r = R\omega, \tag{12b}$$

where $\omega \in \mathbb{R}^{n_\omega}$ and $y_r \in \mathbb{R}^p$. Our goal is to achieve $\lim_{t\to\infty} e_i = 0$ for each $i \in \{1,\dots,N\}$, where e_i is now defined as

$$e_i := y_i - y_r.$$

By the same argument as in Section 3.1, we assume without loss of generality that (S,R) is observable and that all the eigenvalues of S are in the closed right-half complex plane.

In order for the agents to follow the reference trajectory, some information must be available to the network about agent outputs relative to the reference trajectory. In particular, let $\mathscr{I} \subset \{1,\dots,N\}$ be a set of indices corresponding to a subset of agents in the network. We assume that each agent $i \in \{1,\dots,N\}$ has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \mathscr{I}, \\ 0, & i \notin \mathscr{I}. \end{cases}$$

That is, each agent in the index set $\mathscr{I}$ knows the difference between its own output and that of the reference trajectory. To proceed with the design, we need to replace Assumption 1 with a slightly modified assumption.

Assumption $1'$ *Every node of $\mathscr{G}$ is a member of a directed tree with the root contained in $\mathscr{I}$. Furthermore, for each $i \in \{1,\dots,N\}$,*

(1) (A_i,B_i) is stabilizable
(2) (A_i,C_i) is observable
(3) (A_i,B_i,C_i,D_i) is right-invertible
(4) (A_i,B_i,C_i,D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of S

We define the matrix $\bar{G} := G + \mathrm{diag}(\iota_1,\dots,\iota_N)$. It then follows from Lemma 7 in Appendix B that all the eigenvalues of $\bar{G}$ are in the open right-half complex plane.

4.1 Control Design

The control design is similar to that in Section 3.2, except that the exosystem now plays the role of agent K, and we carry out three steps for each agent $i \in \{1,\dots,N\}$. In addition to agent i's system matrices (A_i,B_i,C_i,D_i), the information needed to carry out these three steps is as follows:

- the matrices S and R of the exosystem
- a common integer $\bar{n}$ such that $\bar{n} \geq n_i + n_\omega$ for all $i \in \{1,\dots,N\}$ (see footnote [1] on page 4 for a less conservative definition)
- a common matrix $L \in \mathbb{R}^{p\times p\bar{n}}$, freely chosen
- a common high-gain parameter $\varepsilon \in (0,1]$
- a common number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of the matrix $\bar{G}$

Based on this information, the matrices $\mathscr{A}$, $\mathscr{C}$, Ω_ε, $\mathscr{L}_\varepsilon$, and $\mathscr{P}_\varepsilon$ can be defined in the same way as in Section 3.1.

4.1.1 Design Procedure for Agent $i \in \{1,\dots,N\}$

We follow the exact procedure of Steps 1 and 2 in Section 3.2, with $x_K = \omega$, $y_K = y_r$, and $(A_K,C_K) = (S,R)$.[3] This yields a state $\bar{x}_i$ such that the dynamics of the synchronization error e_i is governed by the system (3), with the same properties as those shown in Lemma 2 (with A_K replaced by S). Similar to Lemma 3, we can therefore state the following result.

Lemma 5 *The regulator equations* (7) *are solvable, and the state-feedback controller $u_i = \bar{F}_i\bar{x}_i$, where $\bar{F}_i = [F_i, \Gamma_i - F_i\Pi_i]$ and F_i is chosen such that $A_i + B_iF_i$ is Hurwitz, ensures that* $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - y_r) = 0$.

We continue by constructing an observer. Let χ_i be defined in the same way as in Step 3 of Section 3.2, to obtain the dynamic equations (9). We construct the observer

$$\begin{aligned}\dot{\hat{\chi}}_i &= (\mathscr{A} + \mathscr{L}_i)\hat{\chi}_i + \mathscr{B}_iu_i + \Omega_\varepsilon\mathscr{P}_\varepsilon\mathscr{C}'(\zeta_i - \hat{\zeta}_i) \\ &\quad + \Omega_\varepsilon\mathscr{P}_\varepsilon\mathscr{C}'(\psi_i - \iota_i(\mathscr{C}\hat{\chi}_i + \mathscr{D}_iu_i)),\end{aligned} \tag{13a}$$
$$\hat{\bar{x}}_i = (T_i'T_i)^{-1}T_i'\hat{\chi}_i. \tag{13b}$$

Finally, we define $\eta_i = \mathscr{C}\hat{\chi}_i + \mathscr{D}_iu_i$ and $u_i = \bar{F}_i\hat{\bar{x}}_i$ as before.

The following lemma, which is proven in Appendix A, shows that all the estimation errors vanish asymptotically if the high-gain parameter ε is chosen sufficiently small.

Lemma 6 *There exists an $\varepsilon^* \in (0,1]$ such that, if ε is chosen such that $\varepsilon \in (0,\varepsilon^*]$, then for each $i \in \{1,\dots,N\}$ we have* $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$.

[3] We note that Assumption $1'$ ensures that Properties 1–4 of Assumption 1 now hold for each $i \in \{1,\dots,N\}$, which facilitates the design in Steps 1 and 2.

Based on Lemmas 5 and 6, we can state the following result, which shows that regulated output synchronization is achieved.

Theorem 2 *There exists an $\varepsilon^* \in (0,1]$ such that, if ε is chosen such that $\varepsilon \in (0,\varepsilon^*]$, then for each $i \in \{1,\ldots,N\}$, $\lim_{t\to\infty}(y_i - y_r) = 0$.*

5 Example

We illustrate the results from Section 3 on a network of ten agents. Agents 1 and 2 are composed as the cascade of a second-order oscillator and a single integrator:

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 3, 4, and 5 are double integrators:

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 6, 7, and 8 are single integrators: $A_i = 0$, $B_i = 1$, $C_i = 1$, $D_i = 0$. Finally, agents 9 and 10 are second-order mass-spring-damper systems:

$$A_i = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

The topology of the network is given by the digraph depicted in Fig. 1, which contains multiple directed spanning trees. One of these is rooted at node 2, and we therefore choose $K = 2$ for our design. The real part of the eigenvalues of the matrix $\bar{G}_2$, constructed by removing row and column 2 from the Laplacian of the digraph in Fig. 1, are lower bounded by approximately 0.33. We assume that a bound $\tau = 0.3$ is known during the design process. We also assume that a bound $\bar{n} = 6$ on $n_i + n_2$, $i \in \{1,\ldots,10\} \setminus 2$, is known. The matrix L is chosen as the zero matrix. Following the design procedure in Section 3.2, we set $u_2 = 0$ and proceed with Steps 1–3 for each of the other agents.

For illustrative purposes, we give the details for agent 3. In Step 1, we first compute O_3 as

$$O_3 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \implies q_3 = 1,\ r_3 = 2.$$

We may choose $\Lambda_{3u} = [1,0]'$ and $\Phi_{3u} = [1,0,0]'$, and hence we can set $\Lambda_3 = I_2$ and $\Phi_3 = I_3$. It follows that

$$\bar{x}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_3 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_2.$$

It can be confirmed that the dynamics of $\bar{x}_i$ with output e_i takes the form of (3) with

$$\bar{A}_{312} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{C}_{32} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

In Step 2, the regulator equations (7) are found to have the solution

$$\Pi_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

We select the matrix $F_3 = [-2 - 3]$ to place the poles of $A_3 + B_3F_3$ at -1 and -2. Thus, we obtain the matrix $\bar{F}_3 = [-2,-3,-3,-1]$.

In Step 3 we design the observer according to the procedure in Section 3.2, with the high-gain parameter $\varepsilon = 0.3$. The relevant matrices for the model (9) are

$$\mathscr{A} = \begin{bmatrix} 0 & I_5 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathscr{B}_3 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}', \quad L_3 = \begin{bmatrix} 0 & 0 & 0.5 & 0 & -0.5 & 0 \end{bmatrix}.$$

We perform the same procedure for the other agents. For agent 1, we obtain $q_i = 3$ and $r_i = 0$; for agents 6, 7, and 8, we obtain $q_i = 1$ and $r_i = 2$; and for agents 9 and 10, we obtain $q_i = 0$ and $r_i = 3$. Fig. 2 shows the resulting simulated output for all ten agents.

6 Concluding Remarks

The designs presented in this paper rely on a set of conditions about the agents and the network that are straightforward to verify. However, they are not all strictly necessary. Inspecting the proofs of our results we see, for example, that the condition on the invariant zeros in Assumption 1 (and 1′) is used only in the proof of Lemma 3 (5) to guarantee that no invariant zeros of (A_i, B_i, C_i, D_i) coincide with the eigenvalues of $\bar{A}_{i22}$. Since the eigenvalues of $\bar{A}_{i22}$ are only a subset of the eigenvalues of A_K (S), the quadruple (A_i, B_i, C_i, D_i) can be allowed to contain certain invariant zeros of A_K (S). Indeed, in the special case of identical agents, the matrix $\bar{A}_{i22}$ vanishes, so the condition on the invariant zeros is not needed. Similarly, the condition of right-invertibility is used

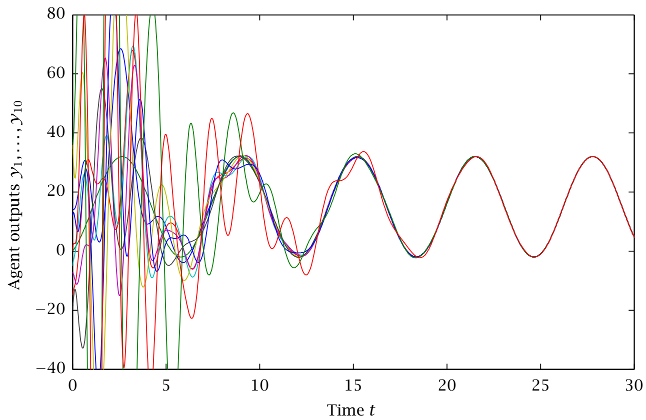

Fig. 2. Outputs from the simulation example

only to guarantee solvability of the regulator equations (7), which vanish for identical agents. Hence, if agent i is identical to A_K, then it does not need to be right-invertible.

Finally, we also note that by choosing $u_K = 0$ and $\eta_K = 0$ in the design for output synchronization, we discard agent K's actuation capability and the information that it receives from the network. It is possible that the assumptions made in this paper can be relaxed by letting all the agents participate actively in the synchronization process (as is done in the regulated output synchronization problem), although this is yet to be investigated. Current research is focused on relaxing the assumptions with respect to right-invertibility and invariant zeros.

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A Proof of Lemmas 1, 2, 3, 4, and 6

PROOF (LEMMA 1) The set of nodes $\{1,\ldots,N\}\setminus K$ can be grouped into directed subgraphs $\mathscr{G}_1,\ldots,\mathscr{G}_M$, each of which has a directed spanning tree rooted at a child of node K. We can assume that there are no edges from $\mathscr{G}_k$ to $\mathscr{G}_j$ if $k>j$ (if such an edge exists, then the child node in $\mathscr{G}_j$ can be moved to $\mathscr{G}_k$). With this permutation, the matrix $\bar{G}_K$ takes the block-triangular form

$$\bar{G}_K = \begin{bmatrix} \tilde{G}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \tilde{G}_{M1} & \cdots & \tilde{G}_{MM} \end{bmatrix}.$$

Each submatrix $\tilde{G}_{ii}$, $i\in 1,\ldots,M$, can be written as $\tilde{G}_{ii} = G_i + D_i$, where G_i is the Laplacian of $\mathscr{G}_i$ and D_i is a diagonal matrix whose j'th entry is the total weight of all the edges to node j of $\mathscr{G}_i$ from nodes in $\mathscr{G}$ outside of $\mathscr{G}_i$. Since $\mathscr{G}_i$ contains a directed spanning tree whose root is a child of node K, the diagonal element in D_i corresponding to that root is positive. It therefore follows from Lemma 7 in Appendix B that all the eigenvalues of $\tilde{G}_{ii}$ are in the open right-half complex plane. The same is true for $\bar{G}_K$, due to its block-triangular form.■

PROOF (LEMMA 2) The definitions of Λ_{iu} and Φ_{iu} imply that the columns of $[\Lambda_{iu}',\Phi_{iu}']'$ span the unobservable subspace of the model (2), which is invariant with respect to blkdiag(A_i,A_K). Hence, there exists a matrix $U_i\in\mathbb{R}^{q_i\times q_i}$ such that

$$\begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix}\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} U_i, \quad \begin{bmatrix} C_i & -C_K \end{bmatrix}\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0. \quad \text{(A.1)}$$

Let $\bar{x}_i$ be partitioned as $\bar{x}_i = [\bar{x}_{i1}',\bar{x}_{i2}']'$, where $\bar{x}_{i1} = x_i - \Lambda_i M_i \Phi_i^{-1} x_K$ and $\bar{x}_{i2} = -N_i\Phi_i^{-1}x_K$. Using the equality $C_i\Lambda_{iu} = C_K\Phi_{iu}$, derived from (A.1), we calculate e_i in terms of $\bar{x}_{i1}$ and $\bar{x}_{i2}$:

$$\begin{aligned} e_i &= C_i x_i - C_K x_K + D_i u_i \\ &= C_i x_i - C_K \begin{bmatrix} \Phi_{iu} & \Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\ &= C_i x_i - \begin{bmatrix} C_i\Lambda_{iu} & C_K\Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\ &= C_i x_i - (C_i\Lambda_i M_i + C_K\Phi_i N_i' N_i)\Phi_i^{-1} x_K + D_i u_i \\ &= C_i(x_i - \Lambda_i M_i\Phi_i^{-1}x_K) - C_K\Phi_i N_i' N_i \Phi_i^{-1} x_K + D_i u_i \\ &= C_i\bar{x}_{i1} + C_K\Phi_i N_i'\bar{x}_{i2} + D_i u_i. \end{aligned}$$

From (A.1), we also have that $A_i\Lambda_{iu} = \Lambda_{iu}U_i$ and $A_K\Phi_{iu} = \Phi_{iu}U_i$. We therefore easily derive that there exist matrices Q_i and R_i on the form

$$Q_i = \begin{bmatrix} U_i & Q_{i12} \\ 0 & Q_{i22} \end{bmatrix}, \quad R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix},$$

such that $A_i\Lambda_i = \Lambda_i Q_i$ and $A_K\Phi_i = \Phi_i R_i$. For $\bar{x}_{i1}$ we can now calculate the state equations as

$$\begin{aligned} \dot{\bar{x}}_{i1} &= A_i x_i - \Lambda_i M_i \Phi_i^{-1} A_K x_K + B_i u_i \\ &= A_i x_i - \Lambda_i M_i R_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i \begin{bmatrix} U_i & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i \begin{bmatrix} U_i & 0 \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K - \Lambda_i \begin{bmatrix} 0 & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i Q_i M_i \Phi_i^{-1} x_K - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i\left(x_i - \Lambda_i M_i \Phi_i^{-1} x_K\right) - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i\bar{x}_{i1} + \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} \bar{x}_{i2} + B_i u_i. \end{aligned}$$

For $\bar{x}_{i2}$ we have $\dot{\bar{x}}_{i2} = -N_i\Phi_i^{-1}A_K x_K = -N_i R_i \Phi_i^{-1} x_K = -R_{i22}N_i\Phi_i^{-1}x_K = R_{i22}\bar{x}_{i2}$. Defining

$$\bar{A}_{i12} = \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix}, \quad \bar{A}_{i22} = R_{i22}, \quad \bar{C}_{i2} = -C_K\Phi_i N_i',$$

we see that e_i is governed by the dynamical equations (3). To see that $(\bar{A}_i,\bar{C}_i)$ is observable, note that the observability matrix O_i of the system (2) has rank n_i+r_i, which is precisely the order of the system (3). To see that the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, note that, due to the block-triangular form of R_i, the eigenvalues of $\bar{A}_{i22} = R_{i22}$ are a subset of the eigenvalues of R_i. Since R_i is obtained from A_K via a similarity transform $R_i = \Phi_i^{-1} A_K \Phi_i$, it has the same eigenvalues as A_K. ■

PROOF (LEMMA 3) Using the notation of the proof of Lemma 2, the task of achieving $\lim_{t\to\infty} e_i = 0$ can be viewed as an output regulation problem, where the subsystem $\dot{\bar{x}}_{i2} = \bar{A}_{i22}\bar{x}_{i2}$ is the exosystem and $\dot{\bar{x}}_{i1} = A_i\bar{x}_{i1} + \bar{A}_{i12}\bar{x}_{i2} + B_i u_i$ is the system to be regulated to achieve $e_i = C_i\bar{x}_{i1} - \bar{C}_{i2}\bar{x}_{i2} + D_i u_i = 0$. Since (A_i, B_i) is stabilizable and the eigenvalues of $\bar{A}_{i22}$ are in the closed right-half complex plane, Saberi et al. (2000, Theorem 2.3.1) shows that the state-feedback controller $u_i = \bar{F}_i\bar{x}_i$ solves the regulation problem, assuming the regulator equations (7) are solvable. From Saberi et al. (2000, Corollary 2.5.1), the regulator equations are solvable if, for each λ that is an eigenvalue of $\bar{A}_{i22}$, the Rosenbrock system matrix $\left[\begin{smallmatrix} A_i-\lambda I & B_i \\ C_i & D_i \end{smallmatrix}\right]$ has rank $n_i + p$. The Rosenbrock system matrix has normal rank $n_i + p$ due to right-invertibility of the quadruple (A_i, B_i, C_i, D_i) (see Saberi, Sannuti, and Chen, 1995, Property 3.1.6). Since this quadruple has no invariant zeros coinciding with eigenvalues of A_K and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, it follows that the rank of the Rosenbrock system matrix is equal to the normal rank for each λ that is an eigenvalue of $\bar{A}_{i22}$. ■

PROOF (LEMMA 4) Let $\tilde{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\begin{aligned}\dot{\tilde{\chi}}_i &= (\mathscr{A} + \mathscr{L}_i)\tilde{\chi}_i - \Omega_\varepsilon \mathscr{P}_\varepsilon \mathscr{C}'(\zeta_i - \hat{\zeta}_i) \\ &= (\mathscr{A} + \mathscr{L})\tilde{\chi}_i - \tilde{\mathscr{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \mathscr{P}_\varepsilon \mathscr{C}'(\zeta_i - \hat{\zeta}_i),\end{aligned}$$

where $\mathscr{L} = [0, L']'$ and $\tilde{\mathscr{L}}_i := \mathscr{L} - \mathscr{L}_i$. Noting that for each $i \in \{1,\ldots,N\}$, $\sum_{j=1}^N g_{ij} = 0$, we have

$$\begin{aligned}\zeta_i &= \sum_{j=1}^N g_{ij} y_j = \sum_{j=1}^N g_{ij}(y_j - y_K) \\ &= \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij} e_j = \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}(\mathscr{C}\chi_j + \mathscr{D}_j u_j).\end{aligned}$$

Also, since $\eta_K = 0$, $\hat{\zeta}_i = \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}(\mathscr{C}\hat{\chi}_j + \mathscr{D}_j u_j)$. It follows that

$$\dot{\tilde{\chi}}_i = (\mathscr{A} + \mathscr{L})\tilde{\chi}_i - \tilde{\mathscr{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}\mathscr{P}_\varepsilon \mathscr{C}'\mathscr{C}\tilde{\chi}_j.$$

Introducing the state transformation $\xi_i = \varepsilon^{-1}\Omega_\varepsilon^{-1}\tilde{\chi}_i$, it can be confirmed that

$$\varepsilon\dot{\xi}_i = (\mathscr{A} + \mathscr{L}_\varepsilon)\xi_i - \tilde{\mathscr{L}}_{i\varepsilon}\xi_i - \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\xi_j,$$

where

$$\tilde{\mathscr{L}}_{i\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{\bar{n}+1}(L - L_i)\Omega_\varepsilon \end{bmatrix}.$$

Define $\xi = [\xi_1', \ldots, \xi_{K-1}', \xi_{K+1}', \ldots, \xi_N']'$ and $\tilde{\mathscr{L}}_\varepsilon = \operatorname{blkdiag}(\tilde{\mathscr{L}}_{1\varepsilon}, \ldots, \tilde{\mathscr{L}}_{(K-1)\varepsilon}, \tilde{\mathscr{L}}_{(K+1)\varepsilon}, \ldots, \tilde{\mathscr{L}}_{N\varepsilon})$, and note that $\|\tilde{\mathscr{L}}_\varepsilon\| = O(\varepsilon)$. The overall dynamics of ξ is

$$\varepsilon\dot{\xi} = (I_{N-1} \otimes (\mathscr{A} + \mathscr{L}_\varepsilon) - \bar{G}_K \otimes (\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}) - \tilde{\mathscr{L}}_\varepsilon)\xi. \quad \text{(A.2)}$$

We shall show that the dynamics in (A.2) can be stabilized by making ε small, in order to diminish $\tilde{\mathscr{L}}_\varepsilon$.

Following the methodology of Wu and Chua (1995b), we define U such that $J = U^{-1}\bar{G}_K U$, where J is the Jordan form of $\bar{G}_K$, and introduce the transformation $\xi = (U \otimes I_{p\bar{n}})\nu$. Then

$$\varepsilon\dot{\nu} = (I_{N-1} \otimes (\mathscr{A} + \mathscr{L}_\varepsilon) - J \otimes (\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}) - \tilde{\mathscr{W}}_\varepsilon)\nu, \quad \text{(A.3)}$$

where $\tilde{\mathscr{W}}_\varepsilon := (U^{-1} \otimes I_{p\bar{n}})\tilde{\mathscr{L}}_\varepsilon(U \otimes I_{p\bar{n}})$. Note that $\|\tilde{\mathscr{W}}_\varepsilon\| = O(\varepsilon)$. Partitioning $\nu = [\nu_1^*, \ldots, \nu_{N-1}^*]^*$ in the same way as ξ, we have that

$$\varepsilon\dot{\nu}_i = \mathscr{R}_i\nu_i - \rho_i\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\nu_{i+1} - \sum_{j=1}^{N-1}\tilde{w}_{\varepsilon ij}\nu_j, \quad i \in 1, \ldots, N-2,$$

$$\varepsilon\dot{\nu}_{N-1} = \mathscr{R}_{N-1}\nu_{N-1} - \sum_{j=1}^{N-1}\tilde{w}_{\varepsilon(N-1)j}\nu_j,$$

where $\mathscr{R}_i = \mathscr{A} + \mathscr{L}_\varepsilon - \lambda_i\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}$; λ_i is the i'th eigenvalue along the diagonal of J; $\rho_i \in \{0,1\}$; and $\tilde{w}_{\varepsilon ij}$ is the (i,j)'th $p\bar{n} \times p\bar{n}$ block of $\tilde{\mathscr{W}}_\varepsilon$. Following the results of Yang et al. (2011c), we can show that $\mathscr{R}_i$ is Hurwitz:

$$\begin{aligned}&\mathscr{R}_i\mathscr{P}_\varepsilon + \mathscr{P}_\varepsilon\mathscr{R}_i^* \\ &= (\mathscr{A} + \mathscr{L}_\varepsilon)\mathscr{P}_\varepsilon + \mathscr{P}_\varepsilon(\mathscr{A} + \mathscr{L}_\varepsilon)' - 2\operatorname{Re}(\lambda_i)\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\mathscr{P}_\varepsilon \\ &= (\mathscr{A} + \mathscr{L}_\varepsilon)\mathscr{P}_\varepsilon + \mathscr{P}_\varepsilon(\mathscr{A} + \mathscr{L}_\varepsilon)' \\ &\quad - 2\tau\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\mathscr{P}_\varepsilon - 2(\operatorname{Re}(\lambda_i) - \tau)\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\mathscr{P}_\varepsilon \leq -I_{p\bar{n}}.\end{aligned}$$

Next, note that there exists an $M_P > 0$ such that for all sufficiently small $\varepsilon > 0$, $\|\mathscr{P}_\varepsilon\| < M_P$. To see this, let $\mathscr{P}$ be the solution of the Riccati equation $\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}' - 2\tau\mathscr{P}\mathscr{C}'\mathscr{C}\mathscr{P} + 2I_{p\bar{n}} = 0$ and let ε be small enough that $\mathscr{L}_\varepsilon\mathscr{P} + \mathscr{P}\mathscr{L}_\varepsilon' \leq I_{p\bar{n}}$. Then clearly $(\mathscr{A} + \mathscr{L}_\varepsilon)\mathscr{P} + \mathscr{P}(\mathscr{A} + \mathscr{L}_\varepsilon)' - 2\tau\mathscr{P}\mathscr{C}'\mathscr{C}\mathscr{P} + I_{\bar{n}p} \leq 0$ and it then follows from standard LQ theory that $\mathscr{P}_\varepsilon \leq \mathscr{P}$ (see, e.g., Kwakernaak and Sivan, 1972).

Define a Lyapunov function $V = \varepsilon\sum_{i=1}^{N-1}\ell_i\nu_i^*\mathscr{P}_\varepsilon^{-1}\nu_i$, where the ℓ_i's are defined recursively by $\ell_{N-1} = 1$ and $\ell_i = \ell_{i+1}/(9M_P^4)$ for $i \in 1, \ldots, N-2$. Then

$$\begin{aligned}\dot{V} = &\sum_{i=1}^{N-1}\ell_i\nu_i^*\mathscr{P}_\varepsilon^{-1}(\mathscr{R}_i\mathscr{P}_\varepsilon + \mathscr{P}_\varepsilon\mathscr{R}_i^*)\mathscr{P}_\varepsilon^{-1}\nu_i \\ &- 2\operatorname{Re}\left(\sum_{i=1}^{N-2}\ell_i\rho_i\nu_i^*\mathscr{P}_\varepsilon^{-1}(\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\mathscr{P}_\varepsilon)\mathscr{P}_\varepsilon^{-1}\nu_{i+1}\right) \\ &- 2\operatorname{Re}\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\ell_i\nu_i^*\mathscr{P}_\varepsilon^{-1}(\tilde{w}_{\varepsilon ij}\mathscr{P}_\varepsilon)\mathscr{P}_\varepsilon^{-1}\nu_j\right)\end{aligned}$$

$$\leq -\sum_{i=1}^{N-1} \ell_i \upsilon_i^2 + 2\sum_{i=1}^{N-2} \ell_i M_P^2 \upsilon_i \upsilon_{i+1} + 2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \ell_i \|\tilde{w}_{\varepsilon ij}\mathscr{P}_\varepsilon\| \upsilon_i \upsilon_j,$$

where $\upsilon_i := \|\mathscr{P}_\varepsilon^{-1}\nu_i\|$. Note that the first two terms can be written as

$$-\frac{1}{3}\sum_{i=1}^{N-1} \ell_i \upsilon_i^2 - \frac{1}{3}\ell_1 \upsilon_1^2 - \frac{1}{3}\ell_{N-1}\upsilon_{N-1}^2 - \sum_{i=1}^{N-2}\left(\frac{\ell_i M_P^2}{\sqrt{\frac{1}{3}\ell_{i+1}}}\upsilon_i - \sqrt{\frac{1}{3}\ell_{i+1}}\upsilon_{i+1}\right)^2 - \sum_{i=1}^{N-2}\left(\frac{1}{3}\ell_i - \frac{\ell_i^2 M_P^4}{\frac{1}{3}\ell_{i+1}}\right)\upsilon_i^2.$$

From the definition of ℓ_i, it can be confirmed that the last term is zero. It follows that $\dot{V} \leq -\frac{1}{3}\sum_{i=1}^{N-1} \ell_i \upsilon_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \ell_i M_P \|\tilde{w}_{\varepsilon ij}\| \upsilon_i \upsilon_j$, Since $\|\tilde{w}_{\varepsilon ij}\| = O(\varepsilon)$ and the ℓ_i's are independent of ε, the first quadratic term dominates the second quadratic term for all sufficiently small ε, and hence $\dot{V}$ is negative definite. It now follows that $\lim_{t\to\infty} \nu = 0$, which implies $\lim_{t\to\infty}\xi = 0$. This in turn implies that $\hat{\chi}_i$ converges to $\chi_i = T_i\bar{x}_i$, and hence $\hat{\bar{x}}_i$ converges to $(T_i'T_i)^{-1}T_i'T_i\bar{x}_i = \bar{x}_i$. ■

PROOF (LEMMA 6) Let $\tilde{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\dot{\tilde{\chi}}_i = (\mathscr{A}+\mathscr{L})\tilde{\chi}_i - \tilde{\mathscr{L}}_i\tilde{\chi}_i - \Omega_\varepsilon\mathscr{P}_\varepsilon\mathscr{C}'(\zeta_i - \hat{\zeta}_i) - \Omega_\varepsilon\mathscr{P}_\varepsilon\mathscr{C}'(\psi_i - \iota_i(\mathscr{C}\hat{\chi}_i + \mathscr{D}_i u_i)),$$

where $\mathscr{L} = [0, L']'$ and $\tilde{\mathscr{L}}_i := \mathscr{L} - \mathscr{L}_i$. Note that

$$\sum_{j=1}^{N} g_{ij}y_j = \sum_{j=1}^{N} g_{ij}(y_j - y_r) = \sum_{j=1}^{N} g_{ij}(\mathscr{C}\chi_j + \mathscr{D}_j u_j).$$

Also, $\hat{\zeta}_i = \sum_{j=1}^{N} g_{ij}(\mathscr{C}\hat{\chi}_j + \mathscr{D}_j u_j)$ and $\psi_i = \iota_i e_i = \iota_i(\mathscr{C}\chi_i + \mathscr{D}_i u_i)$. It follows that

$$\dot{\tilde{\chi}}_i = (\mathscr{A}+\mathscr{L})\tilde{\chi}_i - \tilde{\mathscr{L}}_i\tilde{\chi}_i - \Omega_\varepsilon\left(\sum_{j=1}^{N} g_{ij}\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\tilde{\chi}_j + \iota_i\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\tilde{\chi}_i\right)$$

or, after introducing the state transformation $\xi_i = \varepsilon^{-1}\Omega_\varepsilon^{-1}\tilde{\chi}_i$,

$$\varepsilon\dot{\xi}_i = (\mathscr{A}+\mathscr{L}_\varepsilon)\xi_i - \tilde{\mathscr{L}}_{i\varepsilon}\xi_i - \left(\sum_{j=1}^{N} g_{ij}\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\xi_j + \iota_i\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}\xi_i\right),$$

where $\tilde{\mathscr{L}}_{i\varepsilon}$ is defined in the same way as in the proof of Lemma 4. Defining $\xi = [\xi_1', \ldots, \xi_N']'$ and $\tilde{\mathscr{L}}_\varepsilon = \mathrm{blkdiag}(\tilde{\mathscr{L}}_{1\varepsilon}, \ldots, \tilde{\mathscr{L}}_{N\varepsilon})$, the overall dynamics becomes

$$\varepsilon\dot{\xi} = (I_N \otimes (\mathscr{A}+\mathscr{L}_\varepsilon) - \bar{G}\otimes(\mathscr{P}_\varepsilon\mathscr{C}'\mathscr{C}) - \tilde{\mathscr{L}}_\varepsilon)\xi.$$

The proof can now be completed in the same way as the proof of Lemma 4. ■

B A Useful Lemma

We here give a slightly extended version of Li et al. (2010, Lemma 5).

Lemma 7 *Suppose that $\mathscr{G}$ is a weighted digraph with N nodes, and suppose that $\mathscr{I} \subset \{1, \ldots, N\}$ represents a subset of nodes such that every node of $\mathscr{G}$ is a member of a directed tree with its root contained in $\mathscr{I}$.*[4] *Let G be the Laplacian of $\mathscr{G}$ and let $D = \mathrm{diag}(d_1, \ldots, d_N)$ be a diagonal matrix with non-negative elements. If for each $i \in \mathscr{I}$, $d_i > 0$, then all the eigenvalues of $\bar{G} := G + D$ are in the open right-half complex plane.*

PROOF Let $\hat{\mathscr{G}}$ denote an expanded digraph constructed from $\mathscr{G}$ by adding a node 0 and edges from node 0 to node $i \in \{1, \ldots, N\}$ with weigth d_i, whenever $d_i > 0$. Then the Laplacian of $\hat{\mathscr{G}}$ is given by $\hat{G} = \left[\begin{smallmatrix} 0 & 0 \\ -d & \bar{G}\end{smallmatrix}\right]$, where $d = [d_1, \ldots, d_N]'$. Since $\hat{\mathscr{G}}$ contains edges from 0 to every node in $\mathscr{I}$, it contains a directed spanning tree rooted at node 0. Hence, from Ren and Beard (2005, Lemma 3.3), $\hat{G}$ has a simple eigenvalue at the origin, and all the other eigenvalues are in the open right-half complex plane. Due to the block-triangular form of $\hat{G}$, its eigenvalues consist of the zero element $(1,1)$ and the eigenvalues of $\bar{G}$. It therefore follows that the eigenvalues of $\bar{G}$ must be in the open right-half complex plane. ■

C Auxiliary Model for (A_K, C_K)

Suppose that the model $\dot{x}_K = A_K x_K$, $y_K = C_K x_K$ contains unobservable or asymptotically stable modes. We show here how to construct an observable auxiliary model without asymptotically stable modes, whose output converges to that of the original model. Let Γ_1 be a nonsingular matrix such that the state transformation $\Gamma_1 z_K = x_K$ yields the *stability structural decomposition* (Chen, Lin, and Shamash, 2004)

$$\begin{bmatrix}\dot{z}_{K1} \\ \dot{z}_{K2}\end{bmatrix} = \begin{bmatrix}\hat{A}_{11} & 0 \\ 0 & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}z_{K1} \\ z_{K2}\end{bmatrix}, \quad y_K = \begin{bmatrix}\hat{C}_1 & \hat{C}_2\end{bmatrix}\begin{bmatrix}z_{K1} \\ z_{K2}\end{bmatrix},$$

where $\hat{A}_{11}$ has all its eigenvalues in the closed right-half complex plane and $\hat{A}_{22}$ has all its eigenvalues in the open left-half complex plane. Since z_{K2} vanishes asymptotically, the system $\dot{z}_{K1} = \hat{A}_{11}z_{K1}$, $y_{K1} = \hat{C}_1 z_{K1}$ has the property that $\lim_{t\to\infty}(y_{K1} - y_K) = 0$ for $z_{K1}(0) = [I, 0]\Gamma_1^{-1}x_K(0)$. Next, let Γ_2 be a nonsingular matrix such that the state transformation $\Gamma_2 q_K = z_{K1}$ yields the Kalman observable canonical form:

$$\begin{bmatrix}\dot{q}_{K1} \\ \dot{q}_{K2}\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}q_{K1} \\ q_{K2}\end{bmatrix}, \quad y_{K1} = \begin{bmatrix}0 & \tilde{C}_2\end{bmatrix}\begin{bmatrix}q_{K1} \\ q_{K2}\end{bmatrix}.$$

The reduced-order system $\dot{q}_{K2} = \tilde{A}_{22}q_{K2}$, $y_{K1} = \tilde{C}_2 q_{K2}$ is clearly observable and yields the same output for $q_{K2}(0) = [0, I]\Gamma_2^{-1}z_{K1}(0)$.

[4] A special case is when $\mathscr{I}$ consists of a single element corresponding to the root of a directed spanning tree of $\mathscr{G}$.

D Proof of Column Rank of Λ_{iu} and Φ_{iu}

In this section we demonstrate that the matrices Λ_{iu} and Φ_{iu} must have full column rank. For the sake of establishing a contradiction, suppose that one of the matrices, say Λ_{iu}, has linearly dependent columns. Then there are nonzero vectors $z \in \mathbb{R}^{q_i}$ and $\bar{z} \in \mathbb{R}^{n_K}$ such that

$$\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} z = \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \implies O_i \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} = 0 \implies \begin{bmatrix} C_K \\ \vdots \\ C_K A_K^{n_K-1} \end{bmatrix} \bar{z} = 0.$$

The last statement implies that (A_K, C_K) is unobservable, thus establishing the contradiction.